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A Generalized Universal Matrix Approach for Continuously Inhomogeneous and Curved Finite Elements

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***Abstract*— Finite elements (FE) are often treated as small domains with constant material property tensors (MPT). Recent works have demonstrated the benefits of using a continuously inhomogeneous MPT (CIMPT) in FEs. We are going to show how curvature and CIMPTs can be lumped into a common factor involved in the evaluation FE matrices. The factor carries all the information related to element geometry and MPTs and can be interpolated using an auxiliary basis. This leads to a generalized universal matrix (UM) concept applicable to curved elements with CIMPTs. We demonstrate that the generalized UM method is computationally superior to numerical cubature based evaluation of FE matrices. Integrated with domain decomposition method, the proposed approach is used to solve some practical examples.**

***Index Terms*— generalized universal matrix, continuously inhomogeneous, continuously inhomogeneous material property tensor, curved element, domain decomposition**

I. Introduction

The two common techniques used for evaluation of FE matrices are the universal matrix method and numerical integration that is usually based on some (*Gaussian*) quadrature/cubature rules. Numerical integration is more common in cases where straightforward decomposition of factors of integrations into precomputable look-up tables is not available. For example, numerical quadrature/cubature is often used in presence of element curvature or CIMPTs or when the integrand is not efficiently decomposable into local factors that form a finite dimensional (polynomial) space, e.g. in evaluation of MoM matrices.

Rectilinear geometry and piece-wise constant MPTs are the common assumption in FEM practice. While the assumption of rectilinearilty is often relaxed in curved FE practice [1, 2], the assumption of piece-wise constant MPTs is almost always taken for granted. Nevertheless, physical intuition and practical needs have motivated the use of FEs with CIMPT [3–5]. For example, Webb uses a polynomial representation of magnetic MPT for evaluation of FE matrices arising from a nonlinear magnetic problem [3]. Since the nonlinear magnetic MPT is a function of the magnetic field intensity $\vec{\mathcal{H}}$, it must be able to follow the spatial variations of $\vec{\mathcal{H}}$ that in the very first place were represented by the piece-wise polynomial functions of the FE discretization. This motivates the use of a piece-wise polynomial representation for the magnetic MPT. Ilic et al., use a piece-wise linear representation of MPTs and solve sample problems involving some presumed CIMPT distributions [4]. Recently, in [5], a tensorial formulation is introduced that allows the generalization of the universal matrix concepts into FEs with CIMPTs and curved geometries. Perhaps a difficulty with the tensorial formulation of [5] is that the reader cannot easily relate the presented formulation to the widely used universal matrix method as introduced by Silvester [6]. In this work, however, we shall present a slightly different formulation that is in better harmony with the common understanding of the universal matrix approach. Since the generalized formulation is applicable to FEs with CIMPTs and curvilinear geometries it can be used as an efficient tool in FE implementations.

In what follows, the generalized universal matrix approach is formulated. Then, a simple complexity comparison between the proposed approach and the numerical cubature based method is performed. Finally, in section III, some practical problems are solved using the proposed methodology.

II. Generalized Universal Matrix Method

Systematically speaking, the evaluation of FE matrices can be cast into the following steps [5]:

1) Assume a reference element with a fixed geometry, e.g. Fig. 1a and develop the required polynomial/vector-polynomial basis on the reference element.
2) Develop the required transformation rules for the basis, its various derivations, and the *Jacobian* defined between the reference element and a presumed physical element, e.g. Fig. 1b.
3) As required by the problem weak formulation, express the required integro-differential form first on the physical element.
4) Since, the integro-differential form is originally defined over the physical domain, appropriate transformations must be used to re-express it over the reference element.
5) Identify the factors that are solely determined by the geometry and/or material properties of the physical element from those solely defined (depend) on the reference element. The physical-element geometry dependent factors hereafter referred to as the metric factors.
6) Being independent from the reference element coordinates, the metric factors are pulled out of the integral, while the remaining terms become independent of the metric properties of the physical element. The resulting integrals can hence be precalculated and stored in the

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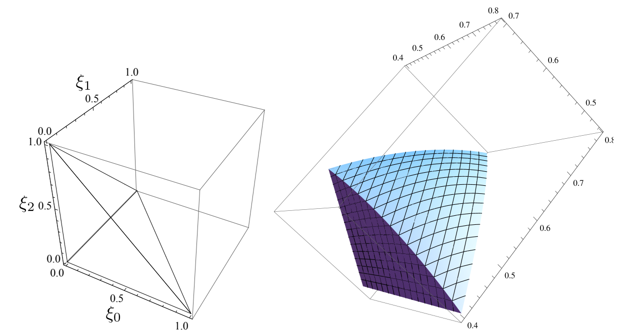

(a) The 3D reference element. (b) An actual curved element.

Fig. 1: A visualization of the concept of reference/physical elements.

so called universal matrices. Consequently the evaluation of FE matrices turns into a sequence of multiply-add operations.

In what follows, we shall implement the abovementioned strategy for evaluation of FE matrices.

A. The Mass Matrix

We shall begin with the derivation of the mass matrix defined in (1).

$$[T]_{n\times n} = \int_{\mathcal{K}_p} \begin{bmatrix} \vec{\lambda}_0 \\ \vec{\lambda}_1 \\ \vdots \\ \vec{\lambda}_{n-1} \end{bmatrix} [\epsilon_r] \begin{bmatrix} \vec{\lambda}_0 & \vec{\lambda}_1 & \dots & \vec{\lambda}_{n-1} \end{bmatrix} d\mathcal{K}_p \quad (1)$$

Through some detailed derivation we show that the following formulation can be used for evaluation of the mass matrix $[T]_{n\times n}$ in which M_s^T are a set of 3×3 matrices (called metric) that carry all the information required to evaluate the mass matrix for an individual physical element.

$$[T]_{n\times n} = \sum_s \int_{\mathcal{K}_r} \begin{bmatrix} \mathcal{F}_0 \\ \mathcal{F}_1 \\ \vdots \\ \mathcal{F}_{n-1} \end{bmatrix} \mathsf{M}_s^T \begin{bmatrix} \mathcal{F}_0 \ \mathcal{F}_1 & \dots & \mathcal{F}_{n-1} \end{bmatrix} \beta_s \xi_0 \xi_1 \xi_2 \quad (2)$$

By looking at (2) we realize that all metric properties of an individual element are lumped in the M_s^T metric. Hence, with respect to each entry of M_s^T, each combination of the other three factors: $\mathcal{F}_i$, β_s and $\mathcal{F}_j$ is a fixed polynomial expression in terms of the barycentric coordinates. Clearly, one can precalculate and store the integral such combinations as an element-metric independent factor. This constitutes to what we call the generalized universal matrix for evaluation of $[T]_{n\times n}$, the mass matrix. Note that in the case of rectilinear elements with element-wise constant MPTs, the $[\mathsf{K}][\epsilon_r][\mathsf{K}]^{\mathsf{T}}\det \mathsf{J}$ factor will be a constant matrix, and the interpolation sum over s runs on one single term. In such a case, the formulation in (2) will be identical to what is known as the conventional universal matrix formulation for the mass matrix $[T]_{n\times n}$.

B. The Stiffness Matrix

The derivation begins with the definition of the stiffness matrix as given in (3).

$$[S]_{n\times n} = \int_{\mathcal{K}_p} \nabla\times \begin{bmatrix} \vec{\lambda}_0 \\ \vec{\lambda}_1 \\ \vdots \\ \vec{\lambda}_{n-1} \end{bmatrix} \frac{1}{[\mu_r]} \nabla\times \begin{bmatrix} \vec{\lambda}_0 & \vec{\lambda}_1 & \dots & \vec{\lambda}_{n-1} \end{bmatrix} d\mathcal{K}_p \quad (3)$$

Similar to the mass matrix, a generalized universal matrix formulation is derived for the stiffness matrix $[S]_{n\times n}$, in which through the use of a metric factor all entities identifying individual physical elements are separated from the others that are solely expressible as polynomials in terms of barycentric variables.

III. Numerical Examples

Our intention in this section it to provide a numerical justification for the proposed generalized universal matrix approach. For this purpose, we shall use the method to solve a few problems with CIMPTs and/or curvilinear boundaries. These will include:

i A dielectric loaded waveguide with CIMPTs [4].
ii Scattering from a *Luneburg* lens.
iii A *Luneburg* lens excited by a waveguide.
iv A conformal spherical PML.

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